The Tank Drainage Problem Revisited: Do These Equations Actually Work?

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The tank drainage problem with pipeline attached is studied in this work. Laminar and turbulent formulations of this unsteady-state flow problem are derived and evaluated by experimental data. Additional literature models are also evaluated for comparison. Several experimental configurations were used including a small tank with a vertical tube, the same with various-sized orifices, a large tank with a horizontal pipe, and a large tank including a piping system with elbows, vertical drop and horizontal extension. Not all the models performed well under all conditions. Limitations of the models are discussed. The model derived by Loiacono and the model we derived (an exact equivalent) showed the best for both laminar and turbulent flow, predicting drainage times to better than ± 8%, on average.

On a étudié dans ce travail le problème du drainage des réservoirs munis de conduites. Les formulations laminaires et turbulentes de ce problème d'écoulement en régime non permanent ont été calculées et évaluées à l'aide de données expérimentales. D'autres modèles venant de la littérature scientifique ont également été évalués à des fins de comparaison. Plusieurs configurations expérimentales ont été utilisées, notamment : un petit réservoir muni d'un tube vertical, le même réservoir comprenant des orifices de tailles diverses, un grand réservoir muni d'une conduite horizontale, un grand réservoir équipé d'un système de conduites ayant des coudes, une chute verticale et une extension horizontale. Tous les modèles n'ont pas donné de bons résultats dans toutes les conditions. Les limites des modèles sont analysées. Le modèle établi par Loiacono et le modèle que nous avons calculé (un équivalent strict) montrent le meilleur potentiel autant pour l'écoulement laminaire que turbulent, prédisant des temps de drainage jusqu'à plus ± 8 % en moyenne.

Keywords: tank drainage, laminar flow, turbulent flow, viscometer correction, exit pipe.

The drainage of a tank by gravity is an old, but knotty problem. The tank may be drained by just a hole (orifice situation) or may be drained through an attached pipe. The pipe may be vertical or horizontal or may include a full piping system with vertical drop and horizontal extension with valves and fittings, etc. The tank usually has a cylindrical shape with a vertical wall, but the bottom may be flat, conical, hemispherical or other shape. Sometimes there is interest in draining the tank completely dry, in which case the bottom shape needs to be accounted for, and sometimes not. In most cases turbulent flow is assumed, and the solutions are useful in plant situations. However, often overlooked is the fact that this geometry is widely used in tube viscometry, in which the laminar flow of a fluid through an attached tube is used to measure its viscosity. Various corrections must be made to get accurate results from an instrument of this type. So the laminar flow case is also important. In some cases the tank is drained by gravity alone; in others, as in the tube viscometer, there is an added pressure head.

When the tank is drained by a hole, Torricelli's equation is used to describe the discharge velocity and flow rate (de Nevers, 1991; Wilkes, 1999; Bird, et al., 2002). This neglects friction losses in the tank and the vena contracta, the contraction of the fluid jet a bit beyond the hole, and is also dependent on turbulent flow or a flat velocity profile. These effects are compensated for by a discharge coefficient, which is usually taken as the orifice coefficient; 0.61 for Reynolds numbers greater than about 10,000, but is sometimes reported as 0.63 (Wilkes, 1999). This value often depends on the Reynolds number and on the type of pressure tap used (for an orifice in a pipe), and the shape of the metal edge defining the orifice hole; for example, radiused, square-edged or sharp-edged can have quite a significant effect (Perry and Greene, 1997). The classical solution for the drainage time is outlined by de Nevers (1991) and Bird, et al. (2002) and appears in many textbooks on fluid mechanics and related fields. This assumes turbulent flow or flat velocity profile of the exiting fluid, but does not include a discharge coefficient and is therefore a solution to a hypothetical problem.

When free drainage (without the use of a pump) is through a pipe (vertical or horizontal) or a pipeline system including elbows, tees, valves, etc. the problem becomes much more complex. This problem has received a lot of attention in the last 15 years. In a recent paper, Keffer (2001) gives the classic analysis method, which is application of the steady-state energy balance equation (sometimes broadly referred to as the Bernoulli equation) coupled with an unsteady-state mass balance.

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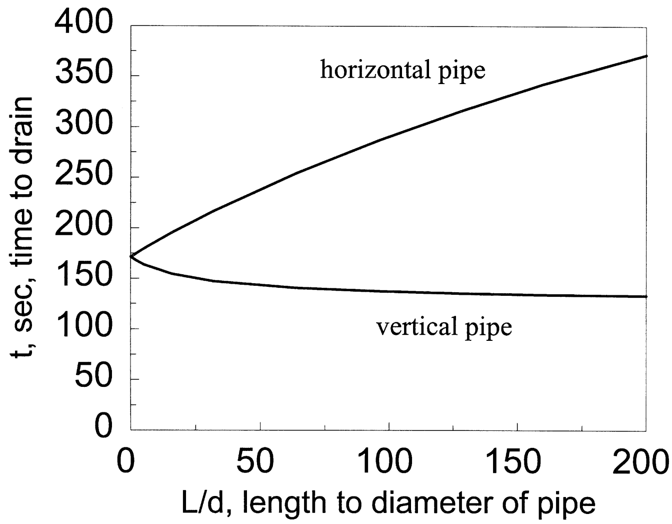

Figure 1. Typical results for tank drainage times (with attached piping) for experimental set-up and dimensions provided by Keffer (2001) for turbulent flow of water, using Equation (3).

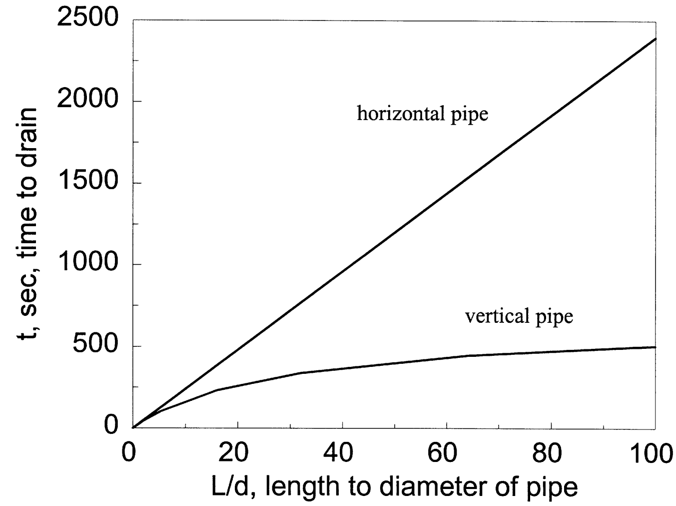

Figure 2. Typical results for tank drainage times (with attached piping) for experimental set-up and dimensions provided by Keffer (2001) for laminar flow of a fluid of 10 cP viscosity, using Equation (5).

If all the correct terms are included, friction losses in particular, this equation is not solvable in closed form, which Keffer points out. A closed form solution requires some simplifying assumptions. The assumptions made by Keffer were not the ones generally used and gave faulty results, but a computer solution was offered that fits his experimental data very well. Experimental data are sorely lacking in the literature, and this has led to other erroneous conclusions about these systems.

Since the head is measured from the surface of the liquid to the tube exit, it will include the pipe length if the tube is vertical rather than horizontal. In the horizontal tube case, the head is independent of the pipe length, whereas in the vertical tube case, the head increases with increasing pipe length. This leads to some interesting effects and underscores the need for proper analytical solutions, as discussed in more detail later on. Whether the flow is laminar or turbulent also has a major impact on the dependence of efflux time on exit pipe length. Figures 1 and 2 illustrate the varieties of behaviour. In the horizontal exit pipe case, the efflux time increases with pipe length for turbulent as well as laminar flow. This is expected due to increase in friction with pipe length. But in the vertical exit pipe case it is surprising to see a decrease in efflux time as length increases for turbulent flow (reaching a minimum as $L \rightarrow \infty$), but not for laminar flow.

The classic development and solution to this problem from a plant engineering perspective is given by Loiacono (1987) for cylindrical flat-bottomed tanks, and added to by Summerfeld and Schwarzhoff (1988) for vessels with hemispherical heads, by Sommerfeld and Shoaei (1989) for tanks with elliptical heads, and by Kossik (2000) for vessels with a conical bottom. Turbulent flow is assumed, and the drain pipe is vertical. Bird, et al (2002) gives solutions for both a spherical tank and a conical tank without piping.

The purpose of this work is to tie all the loose ends together and come up with a comprehensive solution to all these kinds of problems, including the limit of zero pipe length. More importantly, the purpose of this work is to test descriptive equations for tank drainage by experiment, to see which ones actually predict the drainage times well, and which ones don't.

Analytical Models

The classical development of energy balance coupled with unsteady-state mass balance is presented by Bird, et al. (2002), de Nevers (199l), Loiacono (1987), and in somewhat more detailed form by Keffer (2001). Keffer presents a computer method for solution of the differential equation, as does Kossik (2000); both are available on Web-sites listed in the respective papers. Loiacono (1987) gives an analytical form developed by using standard hydrodynamic assumptions. We have developed our own along the lines of Loiacono (1987) but using the resistance coefficient rather than equivalent length.

The standard tank drainage operation is from a vertical, cylindrical tank. The drainage hole may be in the bottom or the side of the tank. The exit pipe may be horizontal, vertical, or may comprise a complex piping system with vertical drop as well as horizontal extension. Or the exit pipe may be very short or an orifice equivalent. The mechanical energy balance can be written in driving force equals resistance format as:

$$\Delta z + \frac{\Delta P}{\rho g} = h_f = \left(\frac{4fL}{d} + \Sigma K' s \right) \frac{v^2}{2g} \tag{1}$$

where Δz is the difference between the liquid level and the outlet of the pipe, ρ is the fluid density, g is the gravitational acceleration, ΔP is the pressure driving force if the tank is pressurized (this term is zero for free drainage), f is the Fanning friction factor, L is the length of straight pipe, d is the pipe diameter (D is the tank diameter), v is the fluid velocity in the pipe or the exit velocity, and K is the resistance coefficient to account for fittings in the line, entrance and exit losses, etc. The pump work term is omitted, since no pump is used, and the velocity of the surface of the fluid in the tank is taken to be negligible in comparison to the velocity of the fluid in the pipe. This is the standard application of the energy balance to this problem.

The (unsteady-state) mass balance can be expressed as:

$$\frac{d}{dt}\left(\frac{\rho\pi D^2}{4}\right)h = -\frac{\rho\pi d^2}{4}v \tag{2}$$

where h is the height of the liquid in the tank.

The solution is obtained by solving Equation (1) for v, substituting this into Equation (2) and solving the differential equation. This is the standard method illustrated in Bird et al. (2002), used by Loiacono (1987) and others. Our solution for the vertical pipe case treats the friction factor as a constant and is:

$$t = \frac{D^2}{d^2}\sqrt{\frac{2\left(\frac{4fL}{d} + \Sigma K's\right)}{g}} \cdot \left(\sqrt{H_i + L_v} - \sqrt{H_f + L_v}\right) \tag{3}$$

where t is the efflux time to drain the tank from fluid height H_i to H_f, and L_v is the vertical drop of the exit pipe. Here it is convenient to define H to be relative to the tank bottom or the centreline of where the exit pipe begins, for example, if the pipe exits horizontally from the tank and not from the bottom. Then Lv is the vertical drop from this point to the pipe exit. For the horizontal pipe case, $L_v = 0$. The assumption of constant friction factor is not a bad one. The value of f can be checked easily by calculating the Reynolds number, and if the average f is different than the initial estimate, that can be used iteratively, if necessary, to get a more correct formula. This was noted by Kossik (2000). The exit kinetic energy loss ($K = 1$) and the entrance loss from tank to pipe ($K = 0.5$) are included in the $\sum K$'s term. These factors would show up in the equivalent-length term of the Loiacono equation, which is the exact equivalent of Equation (3), with L_{eq} substituting for the L term and the $\sum K$'s term dropped. The Loiacono equation is a typical alternative formulation for handling friction losses, but values of Leq are less easily found in the literature than values of resistance coefficient, K, so Equation (3) may be preferred over that given by Loiacono.

If the flow includes a pressure head, the equation reads:

$$t = \frac{D^2}{d^2}\sqrt{\frac{2\left(\frac{4fL}{d} + \Sigma K's\right)}{g}}\left(\sqrt{H_i + L_v + \frac{\Delta P}{\rho g}} - \sqrt{H_f + L_v + \frac{\Delta P}{\rho g}}\right) \tag{4}$$

where the pressure term (assumed to be a constant) shows up inside the square root terms. If the pressure is greater on the outside of the tank than on the inside, thereby opposing the drainage rather than assisting it, the ΔP term would be negative. This is the most general equation.

In laminar flow, the friction factor is not constant, and it is very difficult to estimate the resistance coefficient or friction loss of fittings, contractions, etc. with any degree of accuracy. Some estimates can be found in the Crane Technical Paper No. 410 (1985). If the flow is laminar, we assume all the friction loss term is in the pipe alone, and Bird, et al. (2002) gives a solution which is shown below in our format:

$$t = \frac{32\mu LD^2}{\rho g d^4} \cdot \ln\left(\frac{H_i + L_v}{H_f + L_v}\right) \tag{5}$$

but this neglects the exit kinetic energy and other friction losses in the tank. It may be argued that these are negligible anyway, but that will be tested later. In any case, the exit kinetic energy term is $^1/_2\, v^2$, where the kinetic energy correction factor (Bird, et al., 2002) is used because of the laminar velocity profile. If this term is included in the energy balance, it results in an analytically unsolvable equation. However, Equation (3) can be used for laminar flow as well, if the friction factor is calculated as $16/Re$, where Re is an average Reynolds number for the flow.

If the pipe has zero length (the orifice equivalent), Equation (3) could be used with $L = 0$, or a new equation could be re-derived using the orifice instead of a pipe. In that case, v is given by Torricelli's equation, and the orifice coefficient, C_0, could be used to correct it. Thus:

$$t = \frac{D^2}{d^2 C_0}\sqrt{\frac{2}{g}}\left(\sqrt{H_i} - \sqrt{H_f}\right) \tag{6}$$

In all these cases, analytical solutions can be obtained by making reasonable simplifying assumptions. Figures 1 and 2 show typical results of predicted efflux time from a tank with piping attached (using Equation (3) for turbulent flow in Figure 1, and using Equation (5) for laminar flow in Figure 2) for a tank drainage dimension and situation given by Keffer (2001). The tank is 15 cm in diameter and is drained from a height of 28 cm to 5 cm . One notices in both cases, the vertical tube drains quite a bit faster than the horizontal, because the head increases as the pipe length increases, however there is a limiting flow in both laminar and turbulent situations as $L/d \to \infty$ for the vertical tube case. These limits can be found by taking dt/dL as $L \to \infty$. This is not so simple analytically, but can be done easily numerically. In the horizontal pipe case, the head remains constant, and the drainage time continually increases. In the turbulent flow situation (Figure 1), there is a finite drainage time (equivalent to an orifice value) at $L/d = 0$, but in the laminar (Figure 2) situation there is not, because all the friction losses were assumed to be accounted for in the pipe; hence a pipe of zero length has zero friction loss and gives zero drainage time, which cannot be true. Therefore the laminar equations cannot be valid for very short pipes. The laminar flow case for an orifice is complicated by the orifice coefficient being a strong function of Reynolds number, hence there is no analytical solution for this case.

Experimental

In all cases, the tanks were vertical cylindrical tanks with shallow cones on the bottom. Tanks were drained from one level to another (H_i to H_f), never completely emptied. We did experiments with attached pipes in both vertical and horizontal directions, and with orifices and short L/d pipes, and one piping system with vertical drop and horizontal extension containing fittings. We also did some experiments with additional pressurization. Turbulent flow experiments were done with water at room temperature; laminar flow experiments were done with a 98% glycerol/water solution at room temperature.

The procedure was very straightforward. In both small and large tanks, a ball valve was used to shut off flow. The diameter of the large tank was about 37 cm and that of the small one,

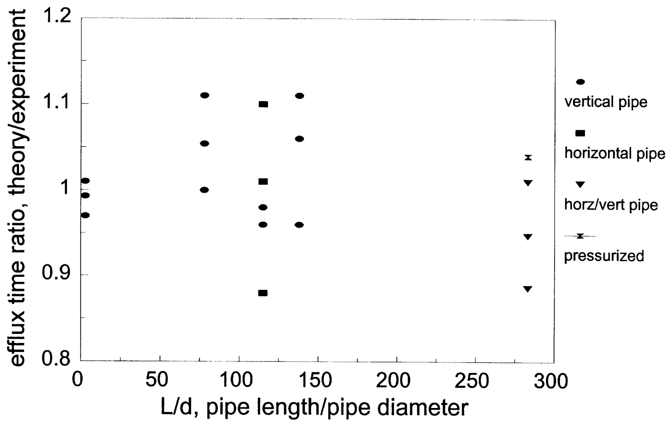

Figure 3. Results of turbulent flow experiments with water at longer *L/d*. Actual efflux times compared to predicted values of Equation (3) or (4). Ratio of 1.0 is perfect agreement.

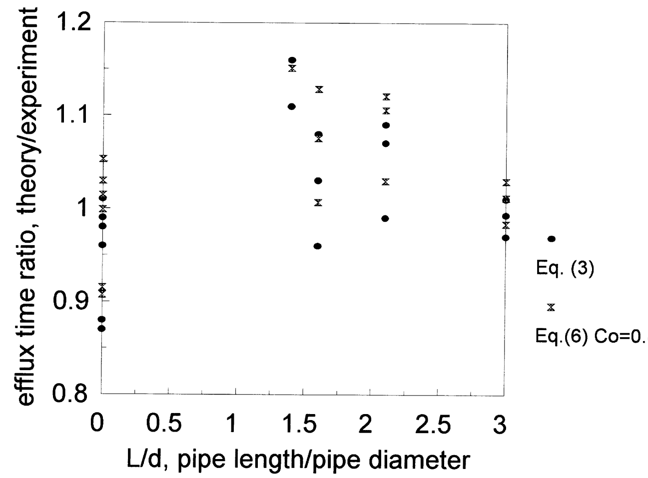

Figure 4. Results of turbulent flow experiments with water at short *L/d*. Actual efflux times compared to predicted values. Ratio of 1.0 is perfect agreement.

7.5 cm. Hard, stainless steel tubing (smooth wall), of diameters ranging from about 0.3 cm to 0.7 cm, was used in the small tank experiments, and hard-drawn copper water pipe (also smooth wall) of diameter 1.9 cm was used with the large tank. Timing was started right after the first liquid exited the tube. Levels were measured with a rule, timing was done with a stopwatch, pressure (when used) was measured by a manometer in the small tank and a calibrated pressure gauge in the large tank.

Results and Discussion

Turbulent Flow Case

The turbulent flow results are shown in Figures 3 and 4, for large *L/d* and short *L/d* respectively and for free drainage. Both figures show that Equation (3) does a very good job of predicting the efflux times for turbulent flow throughout the *L/d* range. The highest deviation is about 15% at only two points out of about 30 in Figure 4, and the average per cent error, computed from the per cent error (defined as the standard deviation/mean) of replicated runs, is less than 8%.

The orifice equation formulation, Equation (6) predicted about 30% high at $L/d = 0$ when an orifice coefficient of 0.61 was used. The tank drainage situation is not exactly the same as the orifice in a pipe, and it may be expected that 0.61 is not the correct coefficient to use. The way the orifice is made has a significant effect on the coefficient. The 0.61 coefficient only applies to sharp-edged orifices with the bevel on the downstream side. If the bevel is upstream, a totally different C_0 must be used. Our orifice matches none of the literature types, being a drilled-out plug with a hole thickness of about 0.1 to 0.2 mm and hole diameters from 0.335 to 0.729 cm. The Marks' Mechanical Engineer's Handbook (1996) recommends a coefficient of 0.80 for short tubes ($L/d \approx 1$), and this is echoed in some of the literature (Schwarzhoff and Sommerfeld, 1998;

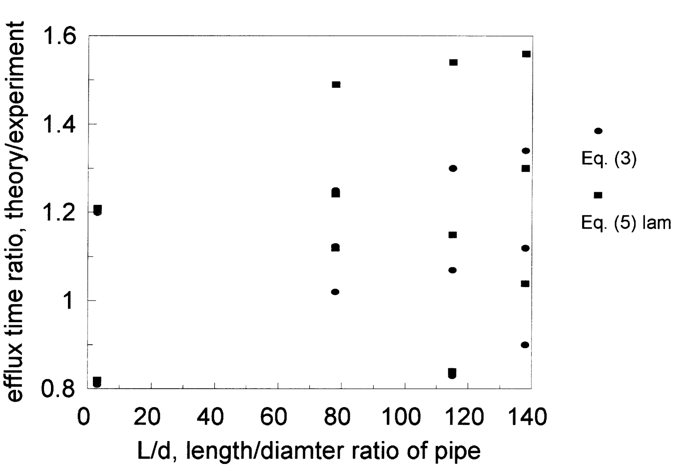

Figure 5. Results of laminar flow experiments with 98% glycerol/water solution at longer *L/d*. Actual efflux times compared to predicted values. Ratio of 1.0 is perfect agreement.

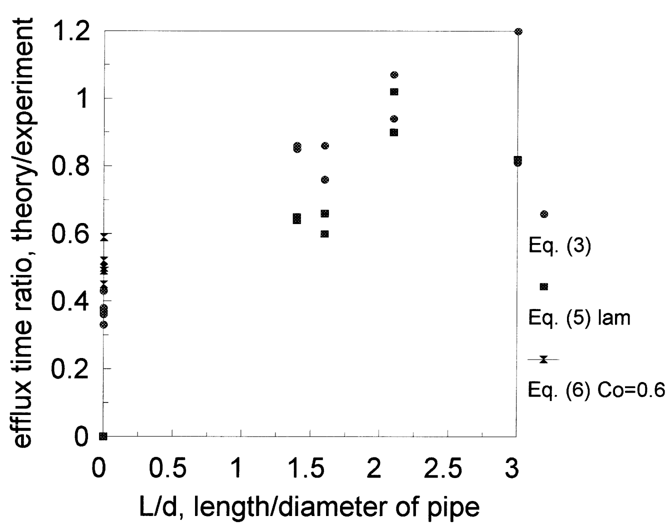

Figure 6. Results of laminar flow experiments with 98% glycerol/water solution at short *L/d*. Actual efflux times compared to predicted values. Ratio of 1.0 is perfect agreement.

Koehler, 1984; Foster, 1981). Using this value improves the fit markedly, bringing the data into almost exact agreement with theory, as shown in the figure. Surprisingly, Equation (3), our equivalent to the Loiacono equation, predicts well at both low and high *L/d*, also including *L/d* = 0. Presumably, it would not predict well for the orifice situation, if the discharge exit was a true sharp-edged orifice.

Added Pressure Head

Experiments were also done with added pressure head in the larger tank, where the pipe system contained mixed horizontal and vertical sections and included fittings. The tank had to be closed to pressurize, and consequently a sight glass had to be used to get a measure of liquid level. We used low pressurization for practical reasons, but also not to dominate the liquid level driving force. Equation (4) predicted the efflux times to better than 5% in our best runs, and to about 15% where we had to make estimates of level.

A short summary of the data for turbulent flow situations follows. Reynolds numbers for the vertical tube ranged from 7,000 to about 18,000; for the horizontal tube from 17,000 to 26,000, and larger pipe with fittings, vertical drop and additional pressure from 20,000 to about 70,000. The largest deviation was about 15% in six occurrences in 90 runs, with an average per cent error, defined as previously, of just slightly less than 8%.

Laminar Flow Case

The laminar flow results are shown in Figures 5 and 6, for large *L/d* and short *L/d* respectively. Both Equation (3), with laminar friction factor, and Equation (5), the laminar equation, were used. In general, Equation (3) does a slightly better job at the larger *L/d*, but both Equation (3) and Equation (5) give essentially the same predictions here.

For short-length tubes, $0 \le L/d \le 3$, the laminar equation is still quite decent at *L/d* = 2, but gets progressively worse and fails completely at *L/d* = 0. This is not surprising, given the formulation of Equation (5). It is surprising that Equation (5) is so good down to an *L/d* ratio as short as two. Equation (3) does better, but it also starts to underpredict at *L/d* = 1.5, and predicts low (not zero) for *L/d* = 0. It does not suffer from the same limitation as the laminar equation, because it includes an energy loss term other than pipe friction. The turbulent flow *K* values cannot be correct for the laminar case, and so it, too, becomes inaccurate when other forms of friction dominate the pipe friction. If appropriate values of *K* could be found or developed, this equation has the potential to predict correctly throughout the range of *L/d* for laminar as well as turbulent flow. However, *K* tends to be a complicated function of Reynolds number and size in these cases, and specific information is scarce.

We also tried the orifice equation, Equation (6), and it, too, is plotted on Figure 6. It gives the best prediction at *L/d* = 0, as might be expected, and is not very good elsewhere, as also might be expected. The orifice equation looks as if it could be used for *L/d* = 0, if the correct C_0 was employed. In the laminar case, C_0 is not constant, does not equal 0.61, and is a complicated function of Reynolds number and hole size. Data matching our exact case here is not available, but using the classic chart provided by Perry's Handbook (1997), one can see that for *Re* = 4, which is typical for the runs we did, the orifice coefficient would be about 0.32, which would bring the prediction into almost exact agreement with the experimentally measured efflux times.

A short summary of the data for laminar flow follows. The Reynolds numbers ranged from 0.07 to about 7.0 for the laminar flow runs. *L/d* ranged from 0 to about 138; experimental data collection times ranged from 5 to about 280 s. Differences in height for the drainage ranged from 1 cm to about 10 cm. The largest deviation was about 32% for threeoccurrences in 36 runs with *L/d* from 1.5 to about 138. The average per cent error was about 14% for both Equation (3) and Equation (5). At *L/d* about 0, no equation predicted well, except the orifice equation with $C_0 = 0.32$ as noted above.

Equation (3) can be used for non-Newtonian fluids, as well, for both laminar and turbulent flows if friction factor can be calculated. For a procedure for calculating the non– Newtonian friction factor using the power-law model see, for example, Perry's Chemical Engineers' Handbook, Section 5-26,27 (1984).

Conclusions

The equation of Loiacono, and its equivalent presented here, does an excellent job of predicting efflux times for drainage of a tank with or without pipe attached, and with or without additional pressure head, with horizontal or vertical exit pipe or some combination of same, with or without fittings, etc. in the line, and for any *L/d*, turbulent or laminar flow (presuming good estimates of *K* or equivalent length can be found for laminar flow situations). This equation should also be valid for non-Newtonian fluids where a friction factor can be calculated.

The efflux time for tank drainage without pipe can be accurately described by incorporating a correct orifice coefficient for both laminar and turbulent flow. This is very dependent on the geometry of the opening. We found that $C_0 = 0.8$, the short tube correction coefficient, to be a very good fit for our thick-edged orifice drilled out from a plug, as well as for short *L/d* up to about 3.0. The commonly used 0.61 orifice coefficient should not be used for anything but an actual, sharp edged orifice with bevel facing downstream.

Nomenclature

C_0	orifice coefficient, dimensionless
d	tube or pipe diameter, (m)
D	tank diameter, (m)
f	Fanning friction factor
g	gravitational acceleration, (m/s^2)
h_f	friction losses, m of fluid
H	height of fluid in tank, (m)
K	resistance coefficient
L	length of straight pipe, (m)
L_{eq}	total equivalent length of pipe, (m)
L_v	vertical drop of pipe, (m)
t	efflux time, (s)
v	fluid velocity, (m/s)
z	height of fluid in tank, (m)

Greek Symbols

Δ_P	pressure difference, inside to outside of tank, (Pa)
μ	fluid viscosity, (Pa·s)
ρ	fluid density, (kg/m3)

Subscripts

f	final conditions
i	initial conditions

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